APPLICATION OF THE RESIDUE THEORY TO TRANSFORMATION OF AUTOMATIC CONTROL SYSTEMS EQUATIONS TO CANONIC VARIABLES

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In stability investigations of automatic control systems canonic variables are frequently used; sometimes it is not only necessary to transform the equations to canonic forms but also to know the transformation matrix.

Lur'e [1] developed formulas for transformation of variables for the case of simple roots of the characteristic equation; applications may be found in the paper by Letov [2].

In the present paper we propose a method of constructing the transformation matrix, based on the residue theory and permitting us to determine transformation coefficients for any structure of the roots of the characteristic equation.

1. Let us consider a system of linear homogeneous equations with constant coefficients

$$\dot{x}_i = \sum_{\tau=1}^{n} a_{i\tau} x_\tau \tag{1.1}$$

The characteristic equation of this system is

$$D(\lambda) = 0 \tag{1.2}$$

and it has nonrepetitive roots $\lambda_1, \ldots, \lambda_n$. Then the general solution of (1.1) may be written in the form [3]

$$x_i = \sum_{\rho=1}^{n} \frac{1}{D'(\lambda_\rho)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda_\rho) \exp \lambda_\rho t \tag{1.3}$$

where $D(\lambda)$, $D_{\mu i}(\lambda)$ are the characteristic determinant and the algebraic cofactors of its elements, respectively.

Let us construct the transformation, the coefficients of which shall be the coefficients of $\exp \lambda_\rho t (\rho = 1, \ldots, n)$ in (1.3),

$$x_i = \sum_{\rho=1}^{n} \frac{1}{D'(\lambda_\rho)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda_\rho) z_\rho \tag{1.4}$$

Let us show that the variables z_i are canonic. To this end let us differentiate (1.4) and substitute into (1.1). In this transformation of the resulting expression let us make use of the obvious equality

$$\sum_{\tau=1}^{n} a_{i\tau} D_{\mu\tau}(\lambda) = \delta_{\mu i} D(\lambda) + \lambda D_{\mu i}(\lambda) \tag{1.5}$$

where $\delta_{\mu i}$ is the Kronecker delta; also let us make use of an expression that directly follows from (1.5) and is given by

$$\sum_{\tau=1}^{n} a_{i\tau} D_{\mu\tau}(\lambda_\rho) = \lambda_\rho D_{\mu i}(\lambda_\rho) \tag{1.6}$$

where λ_ρ is a root of equation (1.2). This will result in the following system of homogeneous algebraic equations with respect to expressions in parenthesis

$$\sum_{\rho=1}^{n} \frac{1}{D'(\lambda_\rho)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda_\rho)(\dot{z}_\rho - \lambda_\rho z_\rho) = 0 \tag{1.7}$$

If the arbitrary constants G_μ are so chosen that no column or row of the determinant of the system (1.7) is zero, then the determinant of the system (1.7) (as the determinant of the fundamental system of solutions) is different from zero. Therefore, the system (1.7) has a unique zero solution, i.e.

$$\dot{z}_\rho = \lambda_\rho z_\rho \qquad (\rho = 1, \ldots, n)$$

It follows from this that the variables z_ρ are canonic. The transformation may be considerably simplified through judicial choice of arbitrary constants C_μ. If the characteristic determinant possesses a row with all algebraic cofactors of its elements different from zero, then it is convenient to take all $C_\mu = 0$ except C_ξ (ξ is the row number); C_ξ is to be chosen such that the transformation coefficients would be as simple as possible.

Let us now assume that there is no such row, i.e. among the algebraic cofactors of the elements of every row there are some equal to zero. Let us select one row with the least number of zero algebraic complements of its elements. Let us denote it by ξ. Then let us take another row with a non-zero algebraic cofactor in the place where the row ξ has its first zero and denote it by η_1. In this fashion let us assign numbers to the rest of the rows $\eta_i = (i = 1, \ldots, k)$ where k does not exceed the number of zero algebraic cofactors in the row ξ. Then it will be convenient to take $C_\xi \neq 0$, $C_{\eta i} \neq 0$ and the remaining $C_\mu = 0$.

2. Let the characteristic equation (1.2) have one root Λ of multiplicity n with respect to the elementary divisors.

As the transformation coefficients for this case, let us use the coefficients of the following functions

$$\frac{t^k}{k!} \exp \Lambda t \qquad (k = 1, \ldots, n)$$

in the general solution written in the form of sum of residues.

Then, taking into account the root structure, we obtain the transformation

$$x_i = \sum_{j=1}^{n} \frac{1}{(n-j)!} \frac{d^{n-j}}{d\lambda^{n-j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^n\right]_{\lambda=\Lambda} z_j \tag{2.1}$$

Let us differentiate (2.1) and substitute the result into (1.1); utilizing (1.5) we then obtain

$$\frac{1}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^n\right]_{\lambda=\Lambda} (\dot{z}_1 - \Lambda z_1) + \tag{2.2}$$

$$+ \sum_{j=2}^{n} \frac{1}{(n-j)!} \frac{d^{n-j}}{d\lambda^{n-j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^n\right]_{\lambda=\Lambda} (\dot{z}_j - z_{j-1} - \Lambda z_j) = 0$$

The determinant of the system (2.2) is different from zero by virtue of of the same considerations as used in the Section 1; consequently, the system has the following unique solution

$$\dot{z}_1 = \Lambda z_1, \qquad \dot{z}_j = z_{j-1} + \Lambda z_j \qquad (j = 2, \ldots, n)$$

From this one can see that z_i are canonic variables.

3. Let the characteristic equation (1.2) have nonrepetitive roots $\lambda_1, \ldots, \lambda_m$ and root Λ of multiplicity k, simple with respect to the elementary divisors ($m + k = n$). Let us form transformation formulas in accordance with the same rules as before and taking into account coefficients of the general solution for the given case of root structure:

$$x_i = \sum_{\rho=1}^{m} \frac{1}{D'(\lambda_\rho)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda_\rho) z_\rho + \sum_{\nu=1}^{n} C_\nu \frac{D_{\nu i}(\Lambda)}{D'(\Lambda)} z_{m+\nu} \tag{3.1}$$

Here the coefficients C_ν are selected among C_μ and are all nonzero. By repeating the above manipulations one can show that the variables z_i are canonic.

4. Let us now assume that (1.2) has a multiple root Λ with the corresponding two groups of solutions, i.e. that the root Λ is repeated twice;

first time it is of multiplicity k_1 with respect to the elementary factors and the second time it is of multiplicity k_2. Let us assume that $k_2 > k_1$. For this case let us write the solution of the system (1.1) in the form of sum of residues as follows

$$x_i = \sum_{j=1}^{k_2} \left\{ \frac{1}{(k_2-j)!} \frac{d^{k_2-j}}{d\lambda^{k_2-j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^{k_2} \right]_{\lambda=\Lambda} \right\} \frac{t^{j-1}}{(j-1)!} \exp \Lambda t \quad (4.1)$$

By selecting C_μ let us divide the solution into two parts: one will contain t^{j-1} ($j = 1, \ldots, k_2$) and the other will contain t^{j-1} ($j = 1, \ldots, k_1$). Taking into account this division the expression within the braces in equation (4.1) will be used as transformation coefficients as follows

$$x_i = \sum_{j=1}^{k_2} \frac{1}{(k_2-j)!} \frac{d^{k_2-j}}{d\lambda^{k_2-j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^{k_2} \right]_{\lambda=\Lambda} z_j + $$
$$+ \sum_{j=1}^{k_1} \frac{1}{(k_2-j)!} \frac{d^{k_2-j}}{d\lambda^{k_2-j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} G_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^{k_2} \right]_{\lambda=\Lambda} z_{k_2+j} \quad (4.2)$$

Constants C_μ and G_μ are subject to the same restrictions as before and, furthermore, let us demand that G_μ will satisfy the following $k_2 - k_1$ conditions

$$\frac{d^{j-1}}{d\lambda^{j-1}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} G_\mu D_{\mu i}(\lambda)(\lambda - \Lambda)^{k_2} \right]_{\lambda=\Lambda} = 0 \qquad (j = 1, \ldots, k_2 - k_1) \quad (4.3)$$

Then the variables z_i will be canonic.

5. Let us now make the most general assumptions regarding the structure of roots of equation (1.2). Let (1.2) have m non-repetitive roots $\lambda_1, \ldots, \lambda_m$. The root λ_α ($\alpha = 1, \ldots, m$) has s_α corresponding groups of solutions, i.e. root λ_α repeats s_α times and at the ith repetition it is of multiplicity k_i^α with respect to the elementary divisors of the following characteristic matrix

$$\left(\sum_{\alpha=1}^{m} \sum_{i=1}^{s_\alpha} k_i^\alpha = n \right)$$

Let us assume that k_i^α are so arranged that $k_i^\alpha \leqslant k_{i+1}^\alpha$ i.e. the number of solutions in a group increases as the group number is increased.

Then the following transformation formulas will be obtained:

$$x_i = \sum_{\alpha=1}^{m} \left\{ \sum_{j=1}^{k_{s\alpha}^\alpha} \frac{1}{(k_{s_\alpha}^\alpha - j)!} \frac{d^{k_{s_\alpha}^\alpha - j}}{d\lambda^{k_{s_\alpha}^\alpha - j}} \left[\frac{1}{D(\lambda)} \sum_{\mu=1}^{n} C_\mu D_{\mu i}(\lambda)(\lambda - \lambda_\alpha)^{k_{s_\alpha}^\alpha} \right]_{\lambda=\lambda_\alpha} z_j^\alpha + \right.$$

$$+\sum_{\beta_\alpha=1}^{s_\alpha-1}\sum_{j=k_{s_\alpha}{}^\alpha-k_{\beta_\alpha}{}^\alpha+1}^{k_{s_\alpha}{}^\alpha}\frac{1}{(2k_{s_\alpha}{}^\alpha-k_{\beta_\alpha}{}^\alpha-j)!}\frac{d^{2k_{s_\alpha}{}^\alpha-k_{\beta_\alpha}{}^\alpha-j}}{d\lambda^{2k_{s_\alpha}{}^\alpha-k_{\beta_\alpha}{}^\alpha-j}}\times$$

$$\times\left[\frac{1}{D(\lambda)}\sum_{\mu=1}^{n}G_\mu{}^{\beta_\alpha}D_{\mu i}(\lambda)(\lambda-\lambda_\alpha)^{k_{s_\alpha}{}^\alpha}\right]_{\lambda=\lambda_\alpha}\Bigg\} \tag{5.1}$$

where C_μ and $G_\mu{}^{\beta_\alpha}$ are so chosen that no row or column of the transformation determinant is equal to zero and, furthermore, constants $G_\mu{}^{\beta_\alpha}$ subject to the following conditions

$$\frac{d^{j-1}}{d\lambda^{j-1}}\left[\frac{1}{D(\lambda)}\sum_{\mu=1}^{n}G_\mu{}^{\beta_\alpha}D_{\mu i}(\lambda)(\lambda-\lambda_\alpha)^{k_{s_\alpha}{}^\alpha}\right]_{\lambda=\lambda_\alpha}=0$$
$$(j=1,\ldots,k_{s_\alpha}{}^\alpha-k_{\beta_\alpha}{}^\alpha;\ \beta_\alpha=1,\ldots,s_\alpha-1;\ \alpha=1,\ldots,m)$$

6. For the system (1.1) one can write inverse transformation formulas expressing canonic variables z_i in terms of the original variables x_j. Liapunov's idea [4] is used in constructing this transformation.

The system, adjoint to (1.1), is of the form

$$y_i=-\sum_{\tau=1}^{n}a_{\tau i}y_i \tag{6.1}$$

The characteristic equation of the system (6.1) has the same root structure as the characteristic equation of (1.2) except that the roots of (6.1) differ in sign from the roots of (1.2). Let us suppose that the roots of the characteristic equation are subject to the same general assumptions as in the Section 5. Then the general solution of the system (6.1) in the form of sum of residues will be written as follows

$$y_i=\sum\frac{1}{\Delta(\lambda)}\sum_{\mu=1}^{n}C_\mu\Delta_{\mu i}(\lambda)\exp\lambda t \tag{6.2}$$

where $\Sigma f(\lambda)$ denotes the sum of residues of the function $f(\lambda)$ at all significant points in a finite region, $\Delta(\lambda)$ and $\Delta_{\mu i}(\lambda)$ are the characteristic determinants of the system (6.1) and the algebraic cofactors of its elements, respectively. As the coefficients of the jth linear inverse transformation form we shall use the elements of the jth column of the coefficient matrix of general solution (6.2), written with the solution groups corresponding to the roots of the characteristic equation. Then the inverse transformation formulas will be as follows:

$$z_j{}^\alpha=\frac{1}{(j-1)!}\sum_{\nu=1}^{n}\frac{d^{j-1}}{d\lambda^{j-1}}\left[\frac{1}{\Delta(\lambda)}\sum_{\mu=1}^{n}C_\mu\Delta_{\mu\nu}(\lambda)(\lambda+\lambda_\alpha)^{k_{s_\alpha}{}^\alpha}\right]_{\lambda=-\lambda_\alpha}x_\nu$$

$$z_{j^*}{}^{\alpha} = \frac{1}{(j-1)!}\sum_{\nu=1}^{n}\frac{d^{j-1}}{d\lambda^{j-1}}\left[\frac{1}{\Delta(\lambda)}\sum_{\mu=1}^{n} G_\mu{}^{\beta_\alpha}\Delta_{\mu\nu}(\lambda)(\lambda+\lambda_\alpha)^{k_{s_\alpha}{}^{\alpha}}\right]_{\lambda=-\lambda_\alpha} x_\nu \quad (6.3)$$

$(j^* = K_1{}^{\alpha} + \ldots + K_{\beta_\alpha}{}^{\alpha} + j;\ \ j = K_{s_\alpha}{}^{\alpha} - K_{\beta_\alpha}{}^{\alpha} + 1, \ldots, K_{s_\alpha}{}^{\alpha};\ \beta_\alpha = 1, \ldots, s_\alpha - 1;\ \alpha = 1, \ldots, m)$

where C_μ and $G_\mu{}^{\beta_\alpha}$ are so chosen that no row or column of the transformation determinant may vanish and, furthermore, that $G_\mu{}^{\beta_\alpha}$ will satisfy the following conditions

$$\frac{d^{j-1}}{d\lambda^{j-1}}\left[\frac{1}{\Delta(\lambda)}\sum_{\mu=1}^{n} G_\mu{}^{\beta_\alpha}\Delta_{\mu\nu}(\lambda)(\lambda+\lambda_\alpha)^{k_{s_\alpha}{}^{\alpha}}\right]_{\lambda=-\lambda_\alpha} = 0 \quad (6.4)$$

$$j = 1, \ldots, k_{s_\alpha}{}^{\alpha} - k_{\beta_\alpha}{}^{\alpha}, \qquad \beta_\alpha = 1, \ldots, s_\alpha - 1;\ \alpha = 1, \ldots, m$$

The fact that the variables z_i determined from (6.3) are canonic may be proved by direct differentiation and utilization of (6.4) for the adjoint system.

Example (see Chetaev [5]).

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -x_1 - 2x_2,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 - 2x_4 + \mu(x_1 + x_2) \quad (6.5)$$

where μ is same parameter.

The characteristic equation of the system (6.5) has a root $\lambda = -1$ of multiplicity four and the elementary divisors will be $(\lambda+1)^3$ and $(\lambda+1)$ [8], i.e. the root $\lambda = -1$ has two corresponding groups of solutions.

The algebraic cofactors of the elements of the first row of the characteristic determinant are all non-zero. Therefore, in the transformation formulas we can take $C_1 = 1$, $C_2 = C_3 = C_4 = 0$. The transformation formulas will be as follows:

$$x_i = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\frac{D_{1i}(\lambda)}{D(\lambda)}(\lambda+1)^3\right]_{\lambda=-1} z_1 + \frac{d}{d\lambda}\left[\frac{D_{1i}(\lambda)}{D(\lambda)}(\lambda+1)^3\right]_{\lambda=-1} z_2 +$$
$$+\left[\frac{D_{1i}(\lambda)}{D(\lambda)}(\lambda+1)^3\right]_{\lambda=-1} z_3 + \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\frac{1}{D(\lambda)}\sum_{\mu=1}^{4} G_\mu D_{\mu i}(\lambda)(\lambda+1)^3\right]_{\lambda=-1} z_4$$

$$(i = 1, 2, 3, 4)$$

where G_μ are subject to the following conditions

$$\frac{d}{d\lambda}\left[\frac{1}{D(\lambda)}\sum_{\mu=1}^{4} G_\mu D_{\mu i}(\lambda)(\lambda+1)^3\right]_{\lambda=-1} = 0, \quad \left[\frac{1}{D(\lambda)}\sum_{\mu=1}^{4} G_\mu D_{\mu i}(\lambda)(\lambda+1)^3\right]_{\lambda=-1} = 0$$

Substituting the algebraic cofactors, we obtain

$$x_1 = -z_1 - z_2 - G_1 z_4,\quad x_3 = -\mu z_3 - G_3 z_4,\quad x_2 = z_2 - G_2 z_4,\quad x_4 = -\mu z_2 + \mu z_3 - G_4 z_4$$

The conditions imposed upon G_μ will assume the form: $G_1 + G_2 = 0$ and $G_3 + G_4 = 0$. Let $G_1 = -G_2 = 1$ and $G_3 = G_4 = 0$. Then, finally, we obtain

$$x_1 = -z_1 - z_2 - z_4, \qquad x_2 = z_2 + z_4, \qquad x_3 = -\mu z_3, \quad x_4 = -\mu z_2 + \mu z_3$$

Let us construct the inverse transformation for the system (6.5). The characteristic equation of the system adjoint to (6.5) will possess a root $\lambda = 1$ of multiplicity four, and of the same structure as the root $\lambda = -1$ in the characteristic equation of the system (6.5).

The algebraic cofactors of the elements of the fourth row of the characteristic determinant are all non-zero; therefore, let $C_4 = 1$, $C_1 = C_2 = C_3 = 0$. For this case the formulas (6.3) will be written as follows:

$$z_k = \frac{1}{(k-1)!} \sum_{\nu=1}^{4} \frac{d^{k-1}}{d\lambda^{k-1}} \left[\frac{\Delta_{4\nu}(\lambda)}{\Delta(\lambda)} (\lambda - 1)^3 \right]_{\lambda=1} x_\nu \qquad (k = 1, 2, 3)$$

$$z_4 = \frac{1}{2!} \sum_{\nu=1}^{4} \frac{d^2}{d\lambda^2} \left[\frac{1}{\Delta(\lambda)} \sum_{\mu=1}^{4} G_\mu \Delta_{\mu\nu}(\lambda)(\lambda - 1)^3 \right]_{\lambda=1} x_\nu$$

where G_μ must satisfy the conditions

$$\left[\frac{1}{\Delta(\lambda)} \sum_{\mu=1}^{4} G_\mu \Delta_{\mu\nu}(\lambda)(\lambda - 1)^3 \right]_{\lambda=1} = 0, \qquad \frac{d}{d\lambda} \left[\frac{1}{\Delta(\lambda)} \sum_{\mu=1}^{4} G_\mu \Delta_{\mu\nu}(\lambda)(\lambda - 1)^3 \right]_{\lambda=1} = 0$$

In accordance with the last formulas we obtain

$$z_1 = \mu x_1 + \mu x_2, \quad z_2 = \mu x_1 + \mu x_2 - x_3 - x_4, \quad z_3 = -x_4, \quad z_4 = -G_1 x_1 - G_2 x_2 - G_3 x_3 - G_4 x_4$$

Here $G_3 = G_4$, $G_1 - G_2 + \mu G_4 = 0$.

Let $G_3 = G_4 = 1$, $G_2 = 0$, $G_1 = -\mu$, then for z_4 we shall have $z_4 = -\mu x_1 - x_3 - x_4$.

7. Let us now consider the equations of a direct automatic control system. Let us suppose that the system is represented by

$$\dot{x}_i = \sum_{\tau=1}^{n} a_{i\tau} x_\tau + h_i f(\sigma), \qquad \sigma = \sum_{\tau=1}^{n} \beta_\tau x_\tau \tag{7.1}$$

where a_{ir}, h_i, β_r are constants, the physical meaning of which has been defined in [1]. Let us construct the corresponding homogeneous system for the system (7.1)

$$\dot{x}_i = \sum_{\tau=1}^{n} a_{i\tau} x_\tau \tag{7.2}$$

It is easy to show that the system (7.1) may be transformed to canonic variables by means of the same transformation as would be used for the homogeneous system (7.2). The transformation for the system (7.2) is already given and is of the form (5.1).

Example (see Popov [6]). Let us consider equations of an aircraft flight correcting control system

$$\dot{x}_1 = -x_1 + f(\sigma), \qquad \dot{x}_2 = -f(\sigma), \qquad \dot{x}_3 = (\gamma - 1)x_1 + \gamma x_2 - rf(\sigma), \qquad \sigma = x_3 \tag{7.3}$$

where γ and r are constants depending on the coefficients of equations describing the elements, the feedback, and the amplifier. These equations are equations of a direct control system, otherwise the equations would be in canonic form. The characteristic equation of the system (7.3) has roots $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 0$ and the multiple root is not simple with respect to the elementary divisors. The algebraic cofactor $D_{12}(\lambda) = 0$ but $D_{22}(\lambda) \neq 0$. Therefore, taking into account assumptions regarding C_μ, let $C_1 = C_2 = 1$, $C_3 = 0$. For the system (7.3) the transformation formulas will be as follows

$$x_i = \frac{D_{1i}(-1) + D_{2i}(-1)}{D'(-1)} z_1 + \left[\frac{d}{d\lambda} \frac{D_{1i}(\lambda) + D_{2i}(\lambda)}{D(\lambda)} \lambda^2\right] z_2 + \left[\frac{D_{1i}(\lambda) + D_{2i}(\lambda)}{D(\lambda)} \lambda^2 z_3\right]_{\lambda=0}$$
$$(i = 1, 2, 3)$$

For this we obtain

$$x_1 = z_1, \qquad x_2 = -z_2, \qquad x_3 = (\gamma - 1)z_1 - (\gamma - 1)z_2 - \gamma z_3$$

and the canonic form of the system is

$$\dot{z}_1 = -z_1 - f(\sigma), \qquad z_2 = f(\sigma)$$
$$\dot{z}_3 = z_2 - \frac{2\gamma - 2 - r}{\gamma} f(\sigma), \qquad \sigma = (\gamma - 1)z_1 - (\gamma - 1)z_2 - \gamma z_3$$

BIBLIOGRAPHY

1. Lur'e, A.I., *Nekotorye nelineinye zadachi teorii avtomaticheskogo regulirovaniia* (*Some Nonlinear Problems of the Automatic Control Theory*). Gostekhizdat, 1951.

2. Letov, A.M., *Ustoichivost' nelineinykh reguliruemykh sistem* (*Stability of Nonlinear Control Systems*). Gostekhizdat, 1955.

3. Smirnov, V.I., *Kurs vysshei matematiki* (*Treatise on Advanced Mathematics*). Gostekhizdat, 1951.

4. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvizheniia* (*General Problem of Stability of Motion*). Gostekhizdat, 1950.

5. Chetaev, N.G., *Ustoichivost' dvizheniia* (*Stability of Motion*). Gostekhizdat, 1955.

6. Popov, E.P., *Dinamika sistem avtomaticheskogo regulirovaniia (Dynamics of Automatic Control Systems)*. Gostekhizdat, 1954.

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